Linear and optimal nonlinear control of one-dimensional maps

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Abstract

We investigate the effects of linear and optimal nonlinear control in the simple case of one-dimensional unimodal maps. We show that linear feedback relates unimodal maps to invertible "Henon-like" maps. This observation should be useful in relating the considerable bodies of knowledge which exist for the two types of systems. In the case of the optimal nonlinear feedback scheme of de Sousa Vieira and Lichtenberg we investigate the relationship between controlled and uncontrolled maps, particularly the preservation of the period doubling route to chaos.

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1 Introduction

In recent years there has been much attention given to theoretical and experimental methods for stabilizing unstable periodic orbits (UPOs) of chaotic systems (see, e.g., [1-11] and refs therein). The methods typically operate by feedback perturbation applied either to an available system parameter [2, 8] or to a state variable [4, 11]. Pyragas [4] suggested delayed feedback control, which incorporates memory into the system. This idea was extended by Socolar, Sukow and Gauthier [9]; and de Sousa Vieira and Lichtenberg [11] (amongst others).

These methods have been the subject of much investigation, both theoretical and experimental, and have proved to be very versatile. An important practical consideration is that feedback can be applied quite simply to physical systems. The

size of the perturbation is determined by a comparison of the current state of the system with the state of the system at τ in the past, where τ is the period of the desired UPO, although the details vary according to the method.

This paper deals with the effect of two different control methods. The first adds a linear perturbation to the system, so that the system $x_{n+1} = f(x_n)$ takes the form $x_{n+1} = f(x_n) + k(x_n - x_{n-1})$. Here $k \in [0, 1)$ is a parameter determining how strongly the system is controlled. This is the return map version of the feedback technique suggested by Pyragas [4], which we will refer to as linear control. The second method adds a nonlinear perturbation, which gives the system the form $x_{n+1} = f(x_n) + k(x_n - f(x_n))$. This is the optimal control method suggested by de Sousa Vieira and Lichtenberg [11].

Both of these methods are capable of successfully stabilizing the fixed points of quite general systems. We show that for some of the unimodal maps on which they have been tested previously they have the effect of producing systems which are already well-known. This observation links these particular controlled systems with others for which there is a body of theoretical understanding.

2 Unimodal Maps with Linear Control

A linearly controlled one-dimensional map can be written as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ as follows

$$x_{n+1} = f(x) + k(x_n - y_n) \tag{1}$$
$$y_{n+1} = x_n \tag{2}$$

where y is used to keep track of the previous iteration. This is similar to what we shall call a *generalized Henon map*

$$X_{n+1} = H(X_n) + Y_n \tag{3}$$
$$Y_{n+1} = BX_n \tag{4}$$
$$H(0) = 1 \qquad \text{(local maximum)} \tag{5}$$

For $B \neq 0$, this generalized Henon map is invertible, with Jacobian B. Thus, those properties of the Henon map which depend on this will generalize.

We shall show that the two systems are equivalent under a simple linear transformation. The most general form giving equivalence of (2) and (4) is

$$\begin{aligned} x &= \alpha BX + \beta \\ y &= \alpha Y + \beta \end{aligned}$$

with

$$B = -k$$

from which

$$H(X) = -\frac{\beta}{\alpha B} - BX + \frac{f(\alpha BX + \beta)}{\alpha B}.$$

The values of α and β are determined by applying the conditions (5).

In the case that $f(x)$ is differentiable, these conditions become $H(0) = 1$, $H'(0) = 0$. For example, from the logistic map

$$f(x) = rx(1-x) \tag{6}$$

we get the standard Henon map [12], for which $H(X) = 1 - AX^2$, with

$$\begin{aligned}
\alpha &= \frac{(k+r)(2+k-r)}{4kr} \\
\beta &= \frac{k+r}{2r} \\
A &= \frac{(k+r)(r-2-k)}{4}
\end{aligned}$$

so the two are in fact equivalent.

The generalized tent map

$$f(x) = \begin{cases} t\dfrac{x}{s} & (0 < x < s) \\ t\dfrac{(1-x)}{(1-s)} & (s < x < 1) \end{cases} \tag{7}$$

is an example where the function f is not differentiable. Applying the conditions (5) gives

$$\alpha = \frac{s-t}{k} \qquad \beta = s$$

so we get a Lozi-style map

$$H(X) = \begin{cases} 1 + \left(k + \dfrac{t}{s}\right)X & (X < 0) \\ 1 + \left(k + \dfrac{t}{s-1}\right)X & (X > 0) \end{cases}$$

The regular Lozi map has the form [13]

$$H(X) = 1 - A|X|$$

Which requires $k + t/s = -k - t/(s-1)$. Thus for a given A and B we have

$$s = \frac{A-B}{2A} \qquad t = \frac{A^2 - B^2}{2A}$$

The interesting point to note from all this is that the often-studied Henon and Lozi maps are in fact standard one-dimensional maps *destabilised* by linear feedback ($k < 0$).

3 Unimodal Maps with Optimal Control

In their paper on nonlinear feedback control [11] de Sousa Vieira and Lichtenberg suggested a control method which combines nonlinear feedback with memory [9], of the form

$$\begin{aligned} x_{n+1} &= f(x_n) + \epsilon_n \\ \epsilon_{n+1} &= -k[f(x_{n+1}) - f(x_n)] + \ell\epsilon_n. \end{aligned}$$

This two-dimensional system works by adding a perturbation based not only on the state of the system at one point in the past, but also on previous perturbations. Here $\ell \in [0,1]$ is an additional parameter which determines the weighting given to the previous perturbation.

They looked for a way to make this system superstable at its fixed point. When $\ell \neq 0$ this occurs only if $\ell = k$, in which case the perturbation ϵ_n becomes entirely dependant on x_n and the system becomes one-dimensional. They referred to this as optimal control.[1] For a one dimensional map $x_{n+1} = f(x_n)$ this optimal control scheme is given by

$$x_{n+1} = (1-k)f(x_n) + kx_n.$$

This method has the advantage that it is very simple and, for the correct choice of k, can make the fixed point not only stable but superstable, regardless of how negative f' is at the fixed point. They applied it to the logistic map, calculating the basins of attraction, upper limit of stability and Lyapunov exponents for particular k and r of this new controlled system. Here we focus on the fact that this form of control is a reduction of dimension.

The optimally controlled logistic map (6) is given by

$$x_{n+1} = (1-k)rx_n(1-x_n) + kx_n. \tag{8}$$

If we make the transformation $x = \alpha X$, where

$$\alpha = \frac{r(1-k)+k}{r(1-k)}$$

it becomes

$$X_{n+1} = RX_n(1-X_n) \tag{9}$$

where $R = r(1-k)+k$. With this transformation, we can determine all the properties of the controlled logistic map from those of the uncontrolled map.

This observation is not limited to the logistic map. In a similar way to the above treatment of linear control, we investigate the circumstances under which a unimodal map with optimal control can be made equivalent to an uncontrolled unimodal map

$$X_{n+1} = F(X_n)$$

[1] Note that here the word does not have its more usual meaning from control theory.

by the simple linear transformation

$$x = \alpha X + \beta.$$

From this

$$F(X) = \frac{(1-k)[f(\alpha X + \beta) - \beta]}{\alpha} + kX$$

and we determine α and β using the unimodal conditions $F(0) = F(1) = 0$. From $F(0) = 0$ we find that β must be a fixed point of f; for simplicity, choose $\beta = 0$. From the condition $F(1) = 0$ we find

$$\frac{f(\alpha)}{\alpha} = \frac{k}{k-1}. \tag{10}$$

Thus, if we can find an α satisfying (10), we can make the two maps equivalent.

As a further example, for the tent map (7) we find

$$\alpha = \frac{t(k-1)}{t(k-1) - k(s-1)}$$

providing that $\alpha > s$. This gives

$$F(X) = \begin{cases} \dfrac{ks + t - tk}{s} X & (0 < X < s/\alpha) \\ \dfrac{(-tk + t - k + ks)}{1-s}(1-X) & (s/\alpha < X < 1) \end{cases}$$

So the optimally controlled tent map it equivalent to an uncontrolled tent map, for suitable combinations of s, t and k.

4 Schwartzian derivative

In [11] de Sousa Vieira and Lichtenberg found that the optimally controlled logistic map undergoes period doubling when its fixed point became unstable. This is only to be expected, since the sytem is equivalent to a rescaled logistic map. Here we wish to investigate the conditions under which period doubling occurs for general unimodal maps which are being stabilised at a fixed point using optimal control. Thus we must examine the relationship between the Schwartzian derivatives SF and Sf, where

$$(Sg)(x) = \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2 \tag{11}$$

The condition for period doubling, at a point where F' decreases through -1, is $SF < 0$ [14]. ($F' = -1$ corresponds to $f' = (1+k)/(k-1)$).

Using $x = \alpha X$ together with (11), we have

$$(SF)(X) = \frac{\alpha^2(1-k)}{[(1-k)f'(x) + k]^2}[(1-k)(f'(x))^2(Sf)(x) + kf'''(x)]$$

Therefore, for $k \in (-1, 1)$, the sign of SF is determined solely by the expression

$$(1-k)(f'(x))^2(Sf)(x) + kf'''(x)$$

which is linear in k, having the same sign as Sf at $k = 0$. The critical observation is that, for small enough k, the period-doubling property is preserved by the application of optimal control. For non-zero k, the rôle of f''' must be considered. For control (rather than destabilisation) $k \in [0, 1)$. If $f''' \leq 0$, we see that period doubling is preserved for all values of k, whereas for $f''' > 0$ the character of the bifurcation may change at some critical value of k.

The relation between Sf and SF in the neighhourhood of a given fixed point is essentially a local property, and does not depend on the requirement that the maps be unimodal. One expects, therefore, that the preservation of the period doubling route to chaos, when a system is subject to optimal control, should be quite general.

5 Conclusion

We have investigated the effects of linear and optimal nonlinear control in the simple case of one-dimensional unimodal maps. Such maps are important testbeds for methods which can also be applied in higher dimensions.

We have shown that the application of linear feedback turns the logistic map into the Henon map; more generally, when applied to any unimodal map, linear feedback results in a "Henon-like" map which is invertible and for which the contraction factor is $|k|$. For example, the linearly controlled tent map is a "Lozi-like" map. It is well-known that the addition of feedback increases the dimensionality of a system; our attention here is particularly on the interconnections which this affords. One might hope to obtain some precise knowledge of strange attractors for the controlled systems in this way. Existing investigations of strange attractors have been for what is, in effect, a linearly destabilised map.

For nonlinear feedback, de Sousa Vieira and Lichtenberg have observed that it is possible to recover the original dimensionality by a procedure which they named optimal control. Having seen that the optimally controlled logistic map is precisely equivalent to an uncontrolled logistic map with a rescaling of parameter, we went on to investigate the relationships which exist for more general unimodal maps, including the tent map. Of particular interest are the conditions under which the inevitable instability of a fixed point (caused by increasing a system parameter) leads into the period doubling route for the controlled system, if this was the route for the original system. In the case of many unimodal maps we have shown that the necessary condition, that the Schwartzian derivative be negative, is preserved for sufficiently small values of feedback parameter $|k|$ (in many cases for all $k \in [0, 1)$).

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